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Instability and degeneracy in the BMN correspondence

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Abstract: Non-degenerate perturbation theory, which was used to calculate the scale dimension of operators on the gauge theory side of the correspondence, breaks down when effects of triple trace operators are included. We interpret this as an instability of excited single-string states in the dual string theory for decay into the continuum of degenerate 3-string states. We apply time-dependent perturbation theory to calculate the decay widths from gauge theory. These widths are new gauge theory data which can be compared with future calculations in light cone string field theory.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The conjectured BMN correspondence [1] potentially gives a perturbative, quantitative match between gauge theory and superstring theory. It was derived from the AdS/CFT correspondence, but it probes string dynamics and thus circumvents the practical limitations of AdS/CFT to the supergravity approximation to string theory. In the BMN correspondence, one considers operators with large $U(1)_R$ charge J. The correlation functions of these operators are governed by two parameters, the effective coupling $\lambda' = g^2_{\rm YM}N/J^2$ and genus expansion parameter $g_2 = J^2/N$, and one works in the limit $N \to \infty, \quad J \to \infty$ with J^2/N fixed.

One family of operators commonly considered are single- and multi-trace operators with two scalar "impurity" fields $\phi(x), \psi(x)$ and J_0 "substrate" fields $Z(x)$,

$$\mathcal{O}^{J_0}_n = \frac{1}{\sqrt{J_0 N^{J_0+2}}} \sum_{l=0}^{J_0} e^{\frac{2\pi i n l}{J_0}} \mathrm{Tr}\left(\phi Z^l \psi Z^{J_0-l}\right) \tag{1.1}$$

and

$$\mathcal{O}^{J_0,J_1,\ldots J_k}_n =: O^{J_0}_n O^{J_1} \cdots O^{J_k} :, \tag{1.2}$$

with $J = J_0 + J_2 + \cdots J_k$. Here O^J is the chiral primary operator,

$$\mathcal{O}^J = \frac{1}{\sqrt{JN^J}} \mathrm{Tr}(Z^J)\,. \tag{1.3}$$

The states of superconformal $\mathcal{N} = 4$ gauge theory on $R \times S^3$ which correspond to these operators[1] are dual to states of the Type IIB superstring quantized in light-cone gauge in the background pp-wave metric and 5-form:

$$\begin{aligned} ds^2 &= -4dx^+dx^- - \mu^2 x^i x^i dx^{+2} + dx^i dx^i\,; \qquad i = 1, \ldots, 8\,, \\ F_{+1234} &= F_{+5678} = \mu \times \mathrm{const}\,. \end{aligned}$$

[1]We use the terms states and operators as if synomymous in the gauge theory.

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The dual string states are those obtained from the ground state $|p^+\rangle$ with light-cone momentum p^+ by the action of creation operators $a^*_\phi(n),\ a^*_\psi(n)$ with world-sheet momentum n. More specifically one considers single- and multi-string states:

$$a^*_\phi(n)a^*_\psi(-n)|p^+\rangle \tag{1.4}$$

and

$$a^*_\phi(n)a^*_\psi(-n)|p^+_0\rangle \otimes |p^+_1\rangle \otimes \cdots |p^+_k\rangle\,, \tag{1.5}$$

where light-cone momenta p^+_i of individual string states are related to corresponding R-charges by $\mu p^+_i\alpha' = \frac{J_i}{\sqrt{g^2_{\rm YM}N}}$. The relation between the parameters of gauge theory and string theory is:

$$\begin{aligned}\lambda' &= \frac{1}{(\mu p^+\alpha')^2}\\ g_2 &= (4\pi g_s)(\mu p^+\alpha')^2\\ g^2_{\rm YM} &= 4\pi g_s\,.\end{aligned} \tag{1.6}$$

Despite the correspondence of notation the string states listed in (1.4) and (1.5) are not the direct maps of the individual operators in (1.1) and (1.2). The reason is that the operators mix through nonplanar graphs [4, 5, 6] even in the free field theory whereas the eigenstates of the free string hamiltonian in (1.5) containing different numbers of strings are orthogonal. An operator S effecting a change of basis in the gauge theory has been identified [2, 3] which makes the gauge theory states in (1.2) orthogonal in the free theory and it is the states obtained by applying S^{-1} to those of (1.2) which map into the string states of (1.5).

Quantitative tests of the correspondence are based on the assumption that the light-cone hamiltonian $P^- = H$ of string theory corresponds to the field theory operator $\Delta - J$, the difference between dilation and R-charge, through the relation

$$\Delta - J = \frac{1}{\mu}H\,. \tag{1.7}$$

In planar order the eigenstates of $\Delta - J$ are the individual states listed in (1.2), and the order g^0_2 eigenvalues are

$$\omega(n, J_0, J_1, \ldots, J_k) = 2 + \lambda'\frac{n^2}{s^2_0} \tag{1.8}$$

with $s_0 = J_0/J$. Operator mixing appears in order g_2, and to this order the k-trace eigenoperator acquires order g_2 admixtures of $(k\pm1)$-trace operators [5, 6]. The eigenvalue is first corrected in order g^2_2 to

$$\omega(n, J_0, J_1, \ldots, J_k) \to 2 + \lambda'\left[\frac{n^2}{s^2_0} + \frac{g^2_2 s^2_0}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32\pi^2n^2}\right)\right]. \tag{1.9}$$

For $k = 1$ the correction was found in [5, 6] and in [7] for $k > 1$. For $k = 0$, the correction exactly matches the genus 1 energy shift of single-string states calculated in light-cone string field theory [8, 9]. This match provides a basis independent test of the relation (1.7). It has also been shown that related matrix elements of H agree with those of $\Delta - J$ after the basis change is made [8, 10].

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The computations of (1.9) in gauge theory used a formalism equivalent to non-degenerate quantum mechanical perturbation theory. Yet it is obvious from (1.8) that the zero order eigenvalues of single-trace operators of momentum n are degenerate with multi-traces of momentum m if $ns_0 = \pm m$. The $n = 1$ state is non-degenerate, since $s_0 = 1$ would be required, and $J_1, J_2, \ldots, J_k$ would vanish. But the $n = 2$ single-trace operator is degenerate with multi-traces with $m = \pm 2$, $s_0 = 1/2$. For $n = 3$ we have degeneracy with multi-traces when $m = \pm 1$, $s_0 = 2/3$ and when $m = \pm 1$, $s_0 = 1/3$, and so forth. One must thus question the validity of non-degenerate perturbation theory, and this was discussed in [6]. One signal for breakdown of perturbation theory is a divergence due to a vanishing energy denominator in the summation formulas for shifts of eigenvalues. It was pointed out in [6] that perturbation theory would remain valid if the matrix elements in the numerator happened to vanish at degeneracy, and that the matrix element of the effective interaction between single- and double-trace states does indeed so vanish. This is a necessary condition for the validity of the order g_2^2 calculation leading to (1.9), but it is not sufficient. The single/triple mixing matrix element is of order g_2^2. If it does not vanish at degeneracy, it would also require [6] the use of degenerate perturbation theory with possible modification of the result (1.9) even though a divergence would not appear until order g_2^4.

An effective quantum mechanical formulation for the gauge theory, which simplifies previous computations was developed in [11]. The single/triple matrix element was computed in this formalism, and it *does not* vanish at degeneracy. It is this fact that motivates the present note in which we discuss the consequences for the physics of the BMN correspondence. In string theory single-string states $|(n, -n)p^+\rangle$ with $n > 1$ would be expected to be unstable, with decay to the continuum of $(k+1)$-string states $|(m, -m)p_0^+, p_1^+, \ldots p_k^+\rangle$ in which the total light cone momentum is divided continuously among the $k + 1$ strings. The lowest case $k = 1$ should correspond to single/double trace mixing in gauge theory, and it has been shown [12] that the relevant string theory matrix element vanishes at order g_2 in agreement with the vanishing gauge theory result mentioned above. Instability would then be expected in string theory via a composite (order g_2^2) process in which the single string first splits into two "virtual" strings and by a further interaction into three final state strings. The non-vanishing single/triple trace mixing matrix element, which is also composite in a sense described below, is the signal of this instability in gauge theory.

More generally the gauge theory has discrete single-trace states $\mathcal{O}_n^J$ embedded in a continuum of multi-traces, since the ratios J_i/J become continuous variables with range 0 to 1 in the BMN limit. It is a well known phenomenon in quantum mechanics that a state which is purely in the discrete sector at time $t = 0$ undergoes irreversible decay to the continuum of states which are degenerate in energy. This phenomenon can be derived using time-dependent perturbation theory [13]. The standard derivation must be generalized in the present case because the relevant process is composite. We make this generalization and compute the order g_2^4 contributions to the energy shift and decay width of single trace states with $n \geq 2$. The energy shift is the less significant datum when there is instability, but it agrees with the principal value calculation of [11]. The value we find for the decay width would also emerge from the formalism of [11] if an $i\epsilon$ prescription had

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been used rather than principal value (as suggested recently in [14]). We show that the decay amplitude is invariant under the basis change discussed above. It should therefore agree with the amplitude computed in string theory.

The development of the continuous spectrum in the string theory is intimately connected with the Penrose limit which produces the pp-wave spacetime from $AdS_5 \times S_5$. For finite radius R of S_5, the null circle in x^- is compact [15] and there is a discrete spectrum of stable states. In the limit $R \to \infty$ the null circle becomes non-compact, producing a continuous spectrum and the possibility of instability.

2. The effective gauge theory hamiltonian

Let us denote an operator in the set (1.2) using the generic notation $\mathcal{O}_i(x)$. With one-loop interactions included, the general form of two-point functions is

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{1}{(x-y)^{2J+4}}[g_{ij} - h_{ij}\ln(x-y)^2 M^2]\,, \tag{2.1}$$

where $g_{ij} = g_{ji}$ is the free-field amplitude which defines a positive-definite inner product or metric, and $h_{ij} = h_{ji}$ describes the order $\lambda' = g^2_{\rm YM}N/J$ interactions. We use a real notation for simplicity, but it is accurate since the correlators of our operators are real. The diagonal elements of g_{ij}, h_{ij} are of order $1+\mathcal{O}(g_2^2)$ while off-diagonal elements between operators containing k and k' traces are of order $g_2^{|k-k'|}$.

One diagonalizes this system via the relative eigenvalue problem

$$h_{ij}v^j_\alpha = \gamma_\alpha g_{ij} v^j_\alpha\,. \tag{2.2}$$

There is a complete set of eigenvectors v^i_α which are orthogonal with respect to the metric, viz. $v^i_\alpha g_{ij} v^j_\beta = \delta_{\alpha\beta}$. The system (2.2) is equivalent to the conventional eigenvalue problem for the "up-down hamiltonian" $h^i_j = g^{ik}h_{kj}$, namely

$$h^i_j v^j_\alpha = \gamma_\alpha v^i_\alpha, \tag{2.3}$$

but one must remember that h^i_j is not naively hermitean (symmetric), but hermitean with respect to g_{ij}, i.e. $g_{ik}h^k_j = h^k_i g_{kj}$.

It is easy to show that the eigenoperators are $\mathcal{O}_\alpha(x) = v^i_\alpha \mathcal{O}_i(x)$ and have diagonal 2-point functions:

$$\begin{aligned}\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(y)\rangle &= \frac{\delta_{\alpha\beta}}{(x-y)^{2J+4}}[1-\gamma_\alpha \ln(x-y)^2M^2] \\ &\sim \frac{\delta_{\alpha\beta}}{(x-y)^{2\Delta_\alpha}}\,,\end{aligned} \tag{2.4}$$

from which one can read the scale dimension $\Delta_\alpha = J+2+\gamma_\alpha$.

A simpler method for field theory computations was developed in [11]. It is a method to compute h^i_j directly with no need for the complicated combinatorics of Feynman diagrams required in earlier work. The resulting effective hamiltonian has a quite simple and striking structure. The method applies to 2-point functions of the operators of (1.2) and has recently been extended [16] to a wider class of scalar operators.

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We refer readers to [11] for an explanation of the method, and we begin our discussion with (11) of that paper. Certain "end-point terms" and other terms which vanish in the BMN limit are neglected in (11), and we note that the omitted terms actually vanish in the channel which is symmetric under exchange $\phi \leftrightarrow \psi$ of the two impurities, so (11) is exact in this channel. Although the BMN limit is taken at an early stage in [11], we use a discrete finite J version of the method and take the limit $J \to \infty$ at the final stage of computation.

In this method h^i_j is replaced by matrix elements of an effective hamiltonian $H = H_0 + H_+ + H_-$. The action of H on gauge theory states/operators is given in (11) of [11] in which the operators contain impurities of fixed spacing l, i.e. $\mathcal{O}_l^{J_0,J_1\ldots J_k} \sim \mathrm{Tr}(\phi \mathrm{Z}^{\mathrm{l}} \psi \mathrm{Z}^{\mathrm{J_0-l}})\mathrm{Tr}\mathrm{Z}^{\mathrm{J_1}} \ldots \mathrm{Tr}\mathrm{Z}^{\mathrm{J_k}}$. After a discrete Fourier transform with respect to l, which is equivalent to that in (1.2) for large J, one obtains the following equations:

$$
\begin{aligned}
H_0 \mathcal{O}_m^{J_0,J_1,\cdots,J_k} &= \lambda' \frac{m^2}{s_0^2} \mathcal{O}_m^{J_0,J_1,\ldots,J_k} \\
H_+ \mathcal{O}_m^{J_0,J_1,\ldots,J_k} &= \lambda' g_2 \sum_{\alpha'_{k+1}} V_{\alpha_k}^{\alpha'_{k+1}} \mathcal{O}_{m'}^{J_0-J_{k+1},J_1,\cdots J_{k+1}} && (2.5) \\
H_- \mathcal{O}_m^{J_0,J_1,\ldots,J_k} &= \lambda' g_2 \sum_{i,\alpha'_{k-1}} V_{\alpha_k}^{\alpha'_{k-1}} \mathcal{O}_{m'}^{J_0+J_i,J_1,\cdots \cancel{J_i},\ldots J_k}\,, && (2.6)
\end{aligned}
$$

where we introduce a collective index notation in the matrix elements $V_{\alpha_k}^{\alpha(k\pm1)}$, namely $\alpha_k = \{m, s_0, \ldots, s_k\}$. Here $s_i = J_i/J$ satisfy $\sum_i s_i = 1$.

The right sides of the equations for H define contributions to matrix elements h^i_j. Symbolically, the structure is $H\mathcal{O}_j = h^i_j \mathcal{O}_i$. Note that the interaction terms are purely of order g_2 and describe the splitting/joining of a $(k+1)$-trace operator into superpositions of $(k+1\pm1)$-trace operators. The hamiltonian $\tilde{H} = SHS^{-1}$ transformed to string basis agrees [14] with the hamiltonian of string bit formalism which contains order g_2 splitting/joining interaction and an order g_2^2 contact term.

For large finite J the matrix elements are

$$
V_{\alpha_k}^{\alpha'_{k+1}} = \frac{1}{\pi^2\sqrt{J}(k+1)!}\sqrt{\frac{s_0 - s'_0}{s_0 s'_0}}\left(\frac{m'}{s'_0}\right)\frac{\sin^2(\pi m s'_0/s_0)}{\frac{m'}{s'_0} - \frac{m}{s_0}}\Delta_{ss'} \tag{2.7}
$$

$$
V_{\alpha_{k+1}}^{\alpha'_k} = \frac{1}{\pi^2\sqrt{J}k!}\sqrt{\frac{s'_0 - s_0}{s_0 s'_0}}\left(\frac{m'}{s'_0}\right)\frac{\sin^2(\pi m' s_0/s'_0)}{\frac{m'}{s'_0} - \frac{m}{s_0}}\Delta_{s's}\,, \tag{2.8}
$$

where $\Delta_{ss'}$ is a product of delta functions,

$$
\Delta_{ss'} = \sum_{P\in S_{k+1}} \delta_{s_1,s'_{P(1)}} \cdots \delta_{s_k,s'_{P(k)}} \delta_{s_0,s'_0+s'_{P(k+1)}}\,. \tag{2.9}
$$

It is in this form that we will use these equations. In the BMN limit, matrix elements we calculate agree with those of the continuum formulation of [11].

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We have also obtained a version of (2.6) valid at any finite J in the $\phi \leftrightarrow \psi$ symmetric channel. Eigenoperators of H_0 are superpositions of those of fixed spacing, namely

$$\mathcal{O}_n^{J_0,J_1\ldots J_k} = \frac{1}{\sqrt{J_0+1}} \sum_{l=0}^{J_0} \cos(\frac{\pi n(2l+1)}{J_0+1}) \mathcal{O}_l^{J_0,J_1\ldots J_k} . \tag{2.10}$$

These operators have eigenvalues of H_0 given by $\frac{g_{\rm YM}^2 N}{\pi^2} \sin^2(\frac{\pi n}{J_0+1})$ which approaches the eigenvalue in (2.6) as $J_0 \to \infty$. These results agree with those of [17, 18]. The interaction terms still describe order g_2 splitting/joining of traces, but they are more complicated than those of (2.6). For example when H_+ acts on the the single-trace $\mathcal{O}_n^J$ one obtains a superposition of double-trace operators $\mathcal{O}_m^{J_0,J_1}$ where $J_0 = J - J_1$ with expansion coefficients

$$\begin{aligned} &\frac{g_{\rm YM}^2 \sqrt{J_1}}{\pi^2 \sqrt{(J+1)(J_0+1)}} \sin\left(\frac{\pi m}{J_0+1}\right) A(n,m,J_1) \\ &A(n,m,J_1) = \frac{1}{2} \sin\left(\frac{\pi n J_1}{J+1}\right) \left\{ \left[\frac{\sin(\pi(\frac{nJ}{J+1} - \frac{m}{J_0+1}))}{\sin(\pi(\frac{n}{J+1} + \frac{m}{J_0+1}))} \right] - [m \to -m] \right\} . \end{aligned} \tag{2.11}$$

One notes that these coefficients exactly reduce to (2.7) in the symmetric channel in the $J \to \infty$ limit. Another thing to note here is that working at finite J does not resolve the degeneracy problem. For example, single- and double-trace operators are degenerate when $\frac{n}{J+1} = \pm \frac{m}{J_0+1}$, and there is a similar degeneracy condition for $(k+1)$-trace operators. One may also observe that $A(n,m,J')$ does not vanish at degeneracy for finite J, although it does vanish as $J \to \infty$. In principle, one should apply degenerate perturbation theory to the calculation of eigenvalues of H even before triple-trace operators are included. However, we will ignore this complication, which very likely disappears in the large J limit.

Our primary goal is to apply time-dependent perturbation theory to study the time evolution of a state which is purely single-trace at time $t = 0$. We will calculate the decay rate of such a state into degenerate triple-trace states. We have found it useful to illustrate the essential physics in quantum mechanical toy models and then adapt the results in the models to the BMN limit of the hamiltonian (2.6).

3. Quantum mechanical models

Our calculation of the decay rate is based on the treatment of the decay of a discrete state embedded in a continuum in [13]. This treatment needs to be modified for our case, but we first review it to set the basic technique in the context of the present up-down matrix formalism.

We thus consider a quantum mechanical system with a set of discrete states $|n\rangle$ and continuum states $|\alpha\rangle$ where α denote the continuous labels of the state. The hamiltonian is $H = H_0 + V$ where H_0 and V are given by the following matrices:

$$\begin{aligned} H_0 &= \begin{pmatrix} E_i \delta^i_j & 0 \\ 0 & E_\alpha \delta(\alpha-\beta) \end{pmatrix}, \\ V &= \begin{pmatrix} 0 & V^i_\alpha \\ V_i^\alpha & 0 \end{pmatrix}. \end{aligned} \tag{3.1}$$

Note that V has vanishing matrix elements between pairs of discrete or pairs of continuum states in agreement with the interactions $H_\pm$ of (2.6). We look for the solution of the Schrödinger evolution equation $i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$, with $|\Psi(t)\rangle$ given by the formal vector:

$$|\Psi(t)\rangle = \begin{pmatrix} a^i(t)e^{-i\,E_i\,t} \\ \phi^\alpha(t)e^{-i\,E_\alpha\,t} \end{pmatrix}$$

and the initial conditions $a^m(0) = \delta^m_n, \quad \phi^\alpha(0) = 0$.

It is easy to see that the discrete and continuum components of $|\Psi(t)\rangle$ satisfy:

$$\begin{aligned} i\frac{d}{dt}a^n(t) &= \int d\alpha V^i_\alpha \exp i(E_n - E_\alpha)t\ \phi^\alpha(t) \\ i\frac{d}{dt}\phi^\alpha(t) &= \sum_m V^\alpha_m \exp i(E_\alpha - E_m)t\ a^m(t)\,. \end{aligned} \tag{3.2}$$

We integrate the second equation using the initial conditions above and substitute the resulting expression for $\phi^\alpha(t)$ in the first equation, obtaining an equation involving only the discrete components, namely

$$\frac{d}{dt}a^n(t) = -\int d\alpha \sum_m \int_0^t dt' \exp i(E_n - E_\alpha)t\ \exp i(E_\alpha - E_m)t'\ V^n_\alpha V^\alpha_m\ a^m(t')\,. \tag{3.3}$$

Next we separate the energy variable in the continuum measure by writing $d\alpha = dEd\beta\rho(\beta, E)$ and define the matrix

$$K^n_m(E) = \int d\beta\rho(\beta, E)\ V^n_\alpha V^\alpha_m\,. \tag{3.4}$$

We assume that $K^n_m(E)$ is a slowly varying function of energy whose scale of variation is ΔE. Substituting (3.4) in (3.3), we find

$$\frac{d}{dt}a^n(t) = -\int_0^\infty dEK^n_m(E)\ \exp i(E_n - E_\alpha)t \int_0^t dt'\ \exp i(E_\alpha - E_m)t' a^m(t')\,. \tag{3.5}$$

We now follow [13] and make the short-time approximation $a^n(t') \approx 1, \quad a^m(t') \approx 0$ for $m \neq n$ in (3.5). We encounter the well known integral

$$\begin{aligned} \int_0^t dt' \exp i(E_n - E)(t - t') &= \frac{\exp i(E_n - E)t\ - 1}{i(E_n - E)} \\ &\approx \pi\delta(E_n - E) + i\mathcal{P}\left(\frac{1}{E_n - E}\right). \end{aligned} \tag{3.6}$$

The last result is strictly correct in the limit $t \to \infty$, but it is effectively valid within integrals of functions f(E) for times much larger than the inverse scale of variation ΔE.

This approximate treatment of (3.5) thus leads to the result

$$\begin{aligned} a^n(t) &\approx 1 - \left(\frac{\Gamma_n}{2} + i\Delta\omega_n\right)t \\ \Gamma_n &= 2\pi K^n_n(E = E_n) \\ \Delta\omega_n &= \mathcal{P}\int_0^\infty dE \frac{K^n_n(E)}{E_n - E}\,. \end{aligned} \tag{3.7}$$

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The quantities $\Gamma_n,\ \Delta\omega_n$ are interpreted as the decay width and energy shift of the unstable state to lowest order in the interaction V. The approximations made in the derivation are valid under the conditions: $1/\Delta E \ll t \ll 1/\Gamma_n$, i.e. the time t must be long compared to the inverse scale of variation of $K_n^n(E)$ and sufficiently short to justify the short-time approximation to (3.5). Note that it was not necessary to specify a scalar product in Hilbert space. It is necessary to define the measure $d\alpha = dEd\beta\rho(\beta, E)$ explicitly. In our problem this measure is determined by the BMN limit of the discrete formulation.

There are additional checks of the self-consistency of the method. One can show that the components $a^m(t),\quad m \neq n$ satisfy $a^m(t) \sim t^2$ for small t, and that the unitarity constraint $|a^n|^2 + \int d\alpha|\phi^\alpha| = 1$ is satisfied to order t.

Applied to our problem the treatment above gives the result $\Gamma_n = 0$, since $K_n^n(E_n)$ vanishes at degeneracy.[2] This is because triple-trace states enter the dynamics only at higher order in the coupling g_2. The toy model above must be generalized to include this effect. Note that the energy shift $\Delta\omega_n$ in (3.7) agrees with the order g_2^2 contribution to the scale dimension of single-trace BMN operators in (1.9).

The generalized model includes three types of states: i) the discrete "single-trace" $|n\rangle$, ii) continuous "double-trace" $|\alpha_2\rangle$, and iii) continuous "triple-trace" $|\alpha_3\rangle$. The free hamiltonian H_0 is diagonal with energy eigenvalues E_n, E_{α_2}, and E_{α_3}, respectively. The interaction matrix and time-dependent state vector which generalize those of the model above are:

$$V = \begin{pmatrix} 0 & V^i_{\alpha_2} & 0 \\ V_i^{\alpha_2} & 0 & V^{\alpha_2}_{\alpha_3} \\ 0 & V^{\alpha_3}_{\alpha_2} & 0 \end{pmatrix},$$

$$|\Psi(t)\rangle = \begin{pmatrix} a^i(t)e^{-i\,E_i\,t} \\ \phi(\alpha_2, t)e^{-i\,E_{\alpha_2}\,t} \\ \sigma(\alpha_3, t)e^{-i\,E_{\alpha_3}\,t} \end{pmatrix}. \tag{3.8}$$

Again we need the solution of the Schrödinger equation $id/dt|\Psi(t)\rangle = (H_0 + V)|\Psi(t)\rangle$ with initial conditions: $a^m(0) = \delta^m_n,\ \phi^{\alpha_2}(0) = 0, \sigma^{\alpha_3}(0) = 0$. The equations linking the components of $|\Psi(t)\rangle$ are

$$i\frac{d}{dt}a^n(t) = \int \mathrm{d}\alpha_2 V^n_{\alpha_2} e^{iE_{n\alpha_2}t}\ \phi^{\alpha_2}(t) \tag{3.9}$$

$$i\frac{d}{dt}\phi^{\alpha_2}(t) = \sum_m V^{\alpha_2}_m e^{iE_{\alpha_2 m}t}\ a^m(t) + \int \mathrm{d}\alpha_3 V^{\alpha_2}_{\alpha_3}\ e^{iE_{\alpha_2\alpha_3}t}\ \sigma^{\alpha_3}(t) \tag{3.10}$$

$$i\frac{d}{dt}\sigma^{\alpha_3}(t) = \int \mathrm{d}\alpha_2 V^{\alpha_3}_{\alpha_2}\ e^{iE_{\alpha_3\alpha_2}t}\ \phi^{\alpha_2}(t)\,. \tag{3.11}$$

We use the notation $E_{n\alpha_2} = E_n - E_{\alpha_2}$, etc. for differences of energy.

We now wish to process the information in (3.9)–(3.11) to obtain a relation describing the coupling of the discrete components of $|\Psi(t)\rangle$ alone. Rather than the exact equation (3.3) in the simpler model, we will obtain a relation which is accurate to fourth order in the potentials. For this purpose we begin in straightforward fashion to integrate (3.11) obtaining an expression for σ^{α_3} in terms of ϕ^{α_2}. We then substitute this in (3.10) and

[2]It is curious that $K_n^n(E) \leq 0$ because $V_n^{\alpha_2}$ is not hermitean.

integrate that, and substitute the result in the first equation which becomes

$$\begin{aligned}\frac{d}{dt}a^n(t) = &-\int \mathrm{d}\alpha \sum_{m'} \int_0^t dt' e^{iE_{n\alpha_2}t+iE_{\alpha_2 m}t'} \; V^n_\alpha V^\alpha_{m'} \; a^{m'}(t') + \\ &+ i\int \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha'_2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' e^{iE_{n\alpha_2}t+iE_{\alpha_2\alpha_3}t'+iE_{\alpha_3\alpha'_2}t''} V^n_{\alpha_2} V^{\alpha_2}_{\alpha_3} V^{\alpha_3}_{\alpha'_2} \phi^{\alpha'_2}(t'')\,.\end{aligned} \tag{3.12}$$

The next step is to substitute for $\phi^{\alpha'_2}(t'')$ in the equation above the value obtained by integrating the first term of (3.10) with "source" a^m. The term with "source" ϕ^{α_2} is of higher order in the potentials through (3.11) and can be dropped. This gives us the net contribution of triple-trace intermediate states, denoted by

$$\begin{aligned}\frac{d}{dt}a^n(t)_{\text{triple}} = &\sum_m \int \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha'_2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' e^{iE_{n\alpha_2}t+iE_{\alpha_2\alpha_3}t'} \times \\ &\times e^{iE_{\alpha_3\alpha'_2}t''+iE_{\alpha_2 m}t'''} \; V^n_{\alpha_2} V^{\alpha_2}_{\alpha_3} V^{\alpha_3}_{\alpha'_2} V^{\alpha'_2}_m \; a^m(t''')\,.\end{aligned} \tag{3.13}$$

To obtain the contribution of single-trace intermediate states at order g_2^4, we integrate the first term of (3.12) to find an expression for $a^n(t)$ (with $n \to m$). This expression is then reinserted for $a^{m'}(t')$ in (3.12) to obtain the iterated contribution

$$\begin{aligned}\frac{d}{dt}a^n(t)_{\text{single}} = &\sum_{m,m'} \int \mathrm{d}\alpha_2\, \mathrm{d}\alpha'_2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' e^{iE_{n\alpha_2}t+iE_{\alpha_2 m}t'} \times \\ &\times e^{iE_{m\alpha'_2}t''+iE_{\alpha_2 m'}t'''} \; V^n_{\alpha_2} V^{\alpha_2}_m V^m_{\alpha'_2} V^{\alpha'_2}_{m'} \; a^{m'}(t''')\,.\end{aligned} \tag{3.14}$$

The full expression for $\frac{d}{dt}a^n(t)$ to fourth order is

$$\begin{aligned}\frac{d}{dt}a^n(t) = &-\int \mathrm{d}\alpha_2 \sum_{m'} \int_0^t dt' e^{iE_{n\alpha_2}t+iE_{\alpha_2 m}t'} \; V^n_{\alpha_2} V^{\alpha_2}_n + \\ &+ \frac{d}{dt}a^n(t)_{\text{triple}} + \frac{d}{dt}a^n(t)_{\text{single}}\end{aligned} \tag{3.15}$$

The first term is just the short-time approximation to (3.5) in the simple model in different notation. We will not discuss it further since it does not contribute to the fourth order amplitudes of primary concern in the generalized model.

We now make the short-time approximation $a^n(t') \approx 1, \quad a^m(t') \approx 0$ for $m \neq n$ in (3.13) and (3.14). We will present the treatment of the triple-trace part explicitly and then summarize the rather similar steps needed for the single-trace part.

The nested set of time integrals in (3.13) can easily be done, and the result (including the overall factor $\exp i(E_{n\alpha_2})t$) is

$$\frac{e^{iE_{n\alpha_3}t}-1}{E_{\alpha_3 n}E_{\alpha_2\alpha_3}} - \frac{e^{iE_{n\alpha_2}t}-1}{E_{\alpha_2 n}E_{\alpha_2\alpha_3}} + \frac{e^{iE_{n\alpha'_2}t}-e^{iE_{n\alpha_2}t}}{E_{\alpha_2\alpha'_2}E_{\alpha_3\alpha'_2}} + \frac{e^{iE_{n\alpha_2}t}-e^{iE_{n\alpha_3}t}}{E_{\alpha_2\alpha_3}E_{\alpha_3\alpha'_2}}\,. \tag{3.16}$$

In an obvious fashion we subtract and add 1 in the last two terms of (3.16). We thus obtain six terms with the structure $\exp iEt - 1$ divided by energy denominators. In our discussion of the contribution of these six terms to (3.13) in the short-time limit, we need the fact

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that all the interaction matrix elements of the actual problem vanish at degeneracy due to the trigonometric factors in (2.7) and (2.8). We assume the same property in the toy model so that all energy integrals which appear when (3.16) is inserted in (3.13) converge despite the singular denominators.

We apply the $i\epsilon$ prescription of the last line of (3.6) in each of the six terms, multiplying and dividing by the energy denominators which are missing in the last four terms. It is easy to see that all contributions cancel pairwise in third and fourth and in the fifth and sixth terms. In the first and second terms we obtain

$$\frac{d}{dt}a^n(0)_{triple} = -\int \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha_2' V^n_{\alpha_2} V^{\alpha_2}_{\alpha_3} \frac{1}{E_{\alpha_2\alpha_3}} V^{\alpha_3}_{\alpha_2'} V^{\alpha_2'}_n \frac{1}{E_{\alpha_2' n}} \times$$
$$\times \left[\pi\delta\left(E_{n\alpha_3}\right) + i\mathcal{P}\left(\frac{1}{E_{n\alpha_3}}\right) - \delta\left(E_{n\alpha_2}\right) - i\mathcal{P}\left(\frac{1}{E_{n\alpha_2}}\right)\right]. \tag{3.17}$$

We now note that the term with $\delta(E_{n\alpha_2})$ vanishes because $V^n_{\alpha_2} = 0$ at degeneracy. Because of this vanishing one can combine the two principal value terms without ambiguity. The triple-trace contribution to the decay amplitude can then be written as

$$\frac{d}{dt}a^n(0)_{\text{triple}} = -\int d\alpha_3 U^n_{\alpha_3}\left[\pi\delta\left(E_{\alpha_3 n}\right) + i\mathcal{P}\left(\frac{1}{E\alpha_3 n}\right)\right] U^{\alpha_3}_n\,, \tag{3.18}$$

where we have introduced the effective composite interactions which couple single- and triple-traces, namely

$$U^n_{\alpha_3} = \int d\alpha_2 \frac{V^n_{\alpha_2} V^{\alpha_2}_{\alpha_3}}{E_{n\alpha_2}} \tag{3.19}$$
$$U^{\alpha_3}_n = \int d\alpha_2 \frac{V^{\alpha_3}_{\alpha_2} V^{\alpha_2}_n}{E_{n\alpha_2}}\,. \tag{3.20}$$

The single-trace contribution (3.14) can be treated similarly. One makes the short-time replacement $a^m(t''') \to \delta^m_n$. The time integrals are then easily done, but separately for the two cases $m \neq n$ and $m = n$. The contributions of the various terms to $\frac{d}{dt}a^n(t)$ at small t are then analyzed as above. In each case there is one term which contributes at $t = 0$ via the $i\epsilon$ prescription. In each there is a $\delta(E_{n\alpha_2})$ which drops because $V^n_{\alpha_2} = 0$. The contributions of two (non-singular) principal value integrals remain in the final result

$$\frac{d}{dt}a^n(0)_{\text{single}} = i\sum_m U^n_m \frac{1-\delta_{Enm}}{E_{mn}} U^m_n - iU^n_n \int d\alpha_2 \frac{V^n_{\alpha_2} V^{\alpha_2}_n}{E^2_{\alpha_n}} \tag{3.21}$$

which is largely written in terms of the composite interaction

$$U^n_m = \int d\alpha_2 \frac{V^n_{\alpha_2} V^{\alpha_2}_m}{E_{n\alpha_2}}\,. \tag{3.22}$$

We can now interpret the results (3.17,3.21) in terms of (3.7). We find the decay width

$$\Gamma_n = 2\pi \int d\alpha_3 \delta(E_{\alpha_3 n}) U^n_{\alpha_3} U^{\alpha_3}_n \tag{3.23}$$

and energy shift

$$\Delta\omega_n = U_n^n - \sum_m U_m^n \frac{1-\delta_{E_{nm}}}{E_{mn}} U_n^m + U_n^n \int d\alpha_2 \frac{V_{\alpha_2}^n V_n^{\alpha_2}}{E_{\alpha_2 n}} - \int d\alpha_3 U_{\alpha_3}^n \mathcal{P}\left(\frac{1}{E_{\alpha_3 n}}\right) U_n^{\alpha_3} \,. \quad (3.24)$$

We are primarily interested in the decay width, which will be evaluated for the BMN system (2.6), (2.7) in section 5. However, we note that the energy shift agrees with the result of fourth order non-degenerate (time-independent) perturbation theory for any quantum-mechanical system whose state space and interaction structure are the same as the present model as defined in (3.8) and the discussion above it.[3] Of course, the principal value derived here to resolve the divergence in the last term is not present in conventional perturbation theory. The physical interpretation of our result is that the pole of the resolvent operator $1/(H_0 - E)$ at $E = E_n$ due to the discrete state $|n\rangle$ is shifted to $E = E_n + \Delta\omega_n - i\Gamma/2$ in the complex plane by the interaction in $1/(H_0 + V - E)$. The state $|n\rangle$ is unstable in the full theory.

The goal of the hamiltonian formulation of [11] was the calculation of anomalous dimensions of BMN operators. In our opinion the calculations undertaken for this purpose should be revised to incorporate degenerate rather than non-degenerate perturbation theory. One may note that the various contributions to our energy shift (3.24) agree exactly with those of [11, (25)]. So the result there should be interpreted as the real part of the shift of a pole rather than the anomalous dimension of an eigenstate of the dilatation operator.

It is interesting that the present time-dependent treatment provides justification for the recent suggestion in [14] of an S-matrix approach to the BMN system (see also [20]) which would require an $i\epsilon$ prescription in the genus two calculations of [11]. The idea of an S-matrix is fully compatible with our interpretation of the instability of the states $|n\rangle$.

4. Decay rate in string basis

It is our intention to derive a formula for the decay widths which can be compared with future calculations in light-cone string field theory. We must therefore determine the effect of the change to the basis in which gauge theory and string theory calculations should match. It turns out that the result is not changed by the basis transformation (at least to order g_2^4). Basis independence would be a triviality if it were implemented as a standard "change of representation" in quantum mechanics, since matrix elements are invariant. However, the situation is different here because states and operators transform contragrediently. We follow the recent paper [14], although the same result should emerge from the similar formalism of [10].

The metric g_{ij} defined by the free two-point functions of the $\mathcal{O}_i$ of (2.1) can be referred to local frames using the vielbein

$$g_{ij} = e_i^k \delta_{kl} e_j^l \,. \quad (4.1)$$

[3] Most treatments of perturbation theory assume a hermitean hamiltonian, but the standard formulas remain valid when rewritten in terms of h_j^i

We found the usual practice of different fonts for the "frame" and "coordinate" indices confusing in the present application, so we prefer to emphasize that the same physical variables are indexed by both upper and lower indices. The inverse vielbein is denoted by f^i_k, so that e.g. $f^i_k e^k_j = \delta^i_j$. The operators with diagonal free two-point functions are

$$\tilde{\mathcal{O}}_k = f^i_k \mathcal{O}_i \equiv S^{-1}\mathcal{O}_k \,, \tag{4.2}$$

where we have defined the Hilbert space operator S^{-1} by the last equality. Our S^{-1} has exactly the properties of $S^{-1/2}$ in [14]. In particular the string basis hamiltonian is

$$\tilde{H} = SHS^{-1} \tag{4.3}$$

which was shown to be exactly the hamiltonian of the string bit formalism [19, 3] whose order g_2 splitting/joining interaction agrees with string field theory and order g_2^2 contact term also agrees (if a certain truncation of intermediate states is made) [9].

The toy models of section 3 can now be described in a new notation in which the states $\mathcal{O}_i$ and $\tilde{\mathcal{O}}_j$ span different bases of the Hilbert space, the gauge theory basis and the string basis, respectively. The time evolution problems treated in section 3 as models for gauge theory can be described as follows. We found (approximately) the state

$$|\Psi(t)\rangle = \sum_i a^i(t)\mathcal{O}_i \tag{4.4}$$

which evolves with time by the hamiltonian H and satisfies an initial condition $a^i(0) = 1$ for $i = i_0$ and $a^i(0) = 0$ for $i \neq i_0$. From the Schrödinger equation $i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$ we easily derive the component evolution

$$i\frac{d}{dt}a^i(t) = H^i_j a^j(t)\,. \tag{4.5}$$

In string basis we would instead be interested in the state

$$|\tilde{\Psi}(t)\rangle = \sum_i \tilde{a}^i(t)\tilde{\mathcal{O}}_i \tag{4.6}$$

which evolves in time by the hamiltonian $\tilde{H}$ with the initial conditions $\tilde{a}^i(0) = 1$ for $i = i_0$ and $\tilde{a}^i(0) = 0$ for $i \neq i_0$.

We now attempt to relate the time-dependent expansion coefficients $a^i(t)$ and $\tilde{a}^i(t)$. The Schrödinger equation $i\frac{d}{dt}|\tilde{\Psi}(t)\rangle = \tilde{H}|\tilde{\Psi}(t)\rangle$ can be expanded as

$$i\frac{d}{dt}\sum_i \tilde{a}^i(t)S^{-1}\mathcal{O}_i = \sum_m \tilde{a}^i(t)\tilde{H}S^{-1}\mathcal{O}_i\,. \tag{4.7}$$

We apply S to both sides and obtain the component equations

$$\begin{aligned} i\frac{d}{dt}\sum_i \tilde{a}^i(t) &= (S\tilde{H}S^{-1})^i_j\tilde{a}^j(t) \\ &= (S^2HS^{-2})^i_j\tilde{\alpha}^j(t)\,. \end{aligned} \tag{4.8}$$

We now consult [14, section 4] and learn that $S^2 H S^{-2} = H^\dagger$. Thus the string basis components evolve via

$$i \frac{d}{dt} \sum_i \tilde{a}^i(t) = (H^\dagger)^i_j \tilde{a}^j(t), \tag{4.9}$$

a curious and useful fact!

To apply this fact we simply go back to the expressions for the decay amplitude in (3.18)–(3.24) and observe that these results remain unchanged if we replace every matrix element V^i_j of the potential by that of the adjoint $(V^\dagger)^i_j = V^j_i$. Thus the results obtained in section 3 in the gauge theory basis exactly describe the decay amplitude in string basis!

5. Computation of the decay rate

In section 3 we derived a formal expression for the decay rate that we reproduce here,

$$\Gamma_n = 2\pi \int \mathrm{d}\alpha_3 U^n_{\alpha_3} U^{\alpha_3}_n \delta(E_{\alpha_3} - E_n)\,. \tag{5.1}$$

We explain the evaluation of this expression in some detail. Let us begin with the computation of the effective composite interaction matrices $U^n_{\alpha_3}$ and $U^{\alpha_3}_n$. Triple-trace state is labelled as $\alpha_3 = \{m, s_0, s_1\}$. For convenience, let us also define $s_2 = 1 - s_0 - s_1$. Because of the δ-function in (5.1), we need this matrix element only at degeneracy, $E_{\alpha_3} = E_n$, where the computation simplifies and has already appeared in [7] and [11]. Using the perturbation coefficients (2.7) in (3.20) one finds,

$$\begin{aligned} U^{m s_0 s_1}_n &= \frac{\lambda' g_2^2 n}{2\pi^4 J} \sqrt{\frac{s_1 s_2}{s_0}} \int_0^1 \mathrm{d}s_0' \frac{1}{s_0'} (\delta_{s_0', s_0+s_1} + \delta_{s_0', s_0+s_2}) \sin^2(\pi n s_0') \times \\ &\quad \times \sum_{p=-\infty}^{\infty} \frac{p}{s_0'} \frac{\sin^2(\pi p s_0/s_0')}{(n^2 - (p/s_0')^2)(n - p/s_0')^2} \end{aligned} \tag{5.2}$$

The evaluation of the sum in the second line is explained in [7, appendix C]. Finally performing the integral over s_0' one arrives at the result,

$$U^{\alpha_3}_n = -\frac{\lambda' g_2^2}{4\pi^2 J} \sqrt{s_1 s_2 s_0} \sin^2(\pi n s_1)\,. \tag{5.3}$$

A similar computation using (2.8) in (3.19) gives the result[4] $U^n_{\alpha_3} = 2U^{\alpha_3}_n$ when $E_{\alpha_3} = E_n$. Inserting (5.3) into (5.1) one obtains,

$$\Gamma_n = \frac{\lambda' g_2^4}{4\pi^3 J^2} \sum_{m=-\infty}^{\infty} J \int_0^1 \mathrm{d}s_0 \; J \int_0^{1-s_0} \mathrm{d}s_1 \, s_1 \, s_2 \, s_0 \sin^4(\pi n s_1) \delta\left(n^2 - \left(\frac{m}{s_0}\right)^2\right). \tag{5.4}$$

We evaluate this expression for $n > 0$ for simplicity but the final expression will be valid also for $n < 0$. The integral over s_1 can be done analytically with the result,

$$\Gamma_n = \frac{\lambda' g_2^4}{128 n^2 \pi^5} \sum_{m=1}^{\infty} \int_0^1 \mathrm{d}s_0 \, \delta\left(n^2 - \left(\frac{m}{s_0}\right)^2\right) s_0(1-s_0)(15 + 4\pi^2 n^2 (1-s_0)^2). \tag{5.5}$$

[4]The factor of 2 arises from the $(k+1)!$ and $k!$ prefactors in (2.7) and (2.8).

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Let us denote the solutions to the degeneracy condition,

$$n^2 = \frac{m^2}{s_0^2}\,,$$

by $\{m^*,\, s_0^*\}$. One then performs the integral over s_0 using the δ-function,

$$\delta\left(n^2 - \left(\frac{m}{s_0}\right)^2\right) = \frac{m}{2n^3}\delta\left(s_0 - \frac{m}{n}\right).$$

This gives,

$$\Gamma_n = \frac{\lambda' g_2^4}{256\pi^5 n^7}\sum_{m^*=1}^{n-1} m^{*2}(n-m^*)\left(15 + 4\pi^2(n-m^*)^2\right). \tag{5.6}$$

Here range of the sum is set by positive solutions to the degeneracy condition above. For example, when $n = 3$ there are two solutions $\{m^*, s_0^*\} = \{1, 1/3\}$ and $\{2, 2/3\}$. The sum in (5.6) is easily done for general n and one arrives at the final result,

$$\Gamma_n = \frac{\lambda' g_2^4}{3840\pi^3 n^5}(n^2-1)\left(n^2 + 1 + \frac{75}{4\pi^2}\right). \tag{5.7}$$

We observe that decay width vanishes for $n = \pm 1$ and it shrinks as the excitation mode n increases.

6. Discussion

The viewpoint we have taken is somewhat simplified in that we have incorporated the degeneracy of single- and triple-trace operators $\mathcal{O}_n^J$ and $\mathcal{O}_m^{J_0,J_1,J_2}$ when $m = \pm n J_0/J$, but we have ignored further degeneracies, such as that of $\mathcal{O}_m^{J_0,J_1,J_2}$ with the five-trace $\mathcal{O}_p^{J_0',J_1,J_2,J_3,J_4}$ when $p = \pm m J_0'/J_0$. Indeed the state $\mathcal{O}_m^{J_0,J_1,J_2}$ is stable only when $m = \pm 1$, so our calculation gives the true order g_2^2 amplitude for decay of $\mathcal{O}_n^J$ to $\mathcal{O}_{m=\pm 1}^{J_0=J/|n|,J_1,J_2}$. The rate is given by the $m^* = 1$ term in the sum (5.6). For $|n| > 2$ and $|m| > 2$, one must envisage a sequential decay process, e.g.

$$\mathcal{O}_{n=3}^J \to \mathcal{O}_{m=2}^{J_0,J_1,J_2} \to \mathcal{O}_{p=1}^{J_0',J_1,J_2,J_3,J_4}$$

with $J_0/J = 2/3$ and $J_0'/J_0 = 1/2$. For general n, there can be a cascade up to $(2|n| - 1)-$trace states. We have not studied transition amplitudes such as $3 \to 5$ or sequential decays explicitly, but we expect that the amplitude of the $1 \to 3 \to 5$ process is of order g_2^4. Thus we believe that the order g_2^2 decay amplitude we have calculated is meaningful for general n, m and that it can be readily compared with string field theory calculations.

Readers should note that most statements made in this paper about vanishing amplitudes hold for the lowest order contributions only. One expects non-vanishing higher order corrections. For example, the decay amplitude for $1 \to 2$ of order g_2^3 is expected to be non-vanishing at degeneracy, and there should be a non-vanishing $1 \to 4$ amplitude of the same order. On the other hand the elementary order g_2 amplitude $V_{\alpha_3}^{\alpha_4}$ in (2.7) vanishes at degeneracy because it is disconnected, *viz.* the delta functions in (2.9).

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As we stated in the Introduction, the computation of anomalous dimensions of BMN operators requires degenerate perturbation theory with possible modification of the order g_2^2 result (1.9). We now briefly describe our preliminary study of this question in which we work at large finite J. In general there are order g_2^2 transition amplitudes which do not vanish at degeneracy between a single trace operator $\mathcal{O}_n^J$ of momentum n and triple-trace operators with several values of m. The triple-traces in turn mix with 5-trace operators, 5-trace with 7-trace, etc., with termination only at the maximal J-trace level. Note that each "band" of k-trace operators involves a finite fraction of J distinct operators. Nevertheless one can derive the precise statement that to order $\lambda' g_2^2$ the anomalous dimensions of operators in this large set are the eigenvalues of $H_0 + \lambda' g_2^2 U$ where U is the effective composite interaction whose non-vanishing matrix elements are,

$$U_{\alpha_k}^{\alpha'_{k\pm2}} = \int \mathrm{d}\alpha_{k\pm1} \frac{V_{\alpha_{k\pm1}}^{\alpha_{k\pm2}} V_{\alpha_k}^{\alpha_{k\pm1}}}{E_{n\alpha_{k\pm1}}}\,, \qquad U_{\alpha_k}^{\alpha'_k} = \sum_{j=\pm1} \int \mathrm{d}\alpha_{k+j} \frac{V_{\alpha_{k+j}}^{\alpha'_k} V_{\alpha_k}^{\alpha_{k+j}}}{E_{n\alpha_{k+j}}}\,. \tag{6.1}$$

Note that single-triple matrix elements appeared in section 3. For large J this is a very large but sparse matrix.

We have studied a reasonably accurate "model" of this matrix to determine the large J limit of eigenvalues and eigenvectors. The results indicate that there is a single eigenvector which is "mostly single trace" as $g_2^2 \to 0$, and its eigenvalue is that of (1.9) for $k = 0$. Similarly, there are some "mostly k-trace" eigenvectors for which (1.9) is also correct. But there are also some "collective" eigenvectors which are superpositions of many multi-trace operators and whose eigenvalues do not agree with (1.9). Thus our model indicates that (1.9) is correct for most states in the system but not correct for all multi-trace states.

We close by noting that the gauge theory result (1.9) for "mostly k-trace" operators has not yet been confirmed in light cone string field theory. The complication of degenenerate perturbation theory described above in the gauge theory will be mirrored in string theory. Thus we expect that the string theory computation can be organized to produce a composite interaction matrix at degeneracy which should agree with (6.1) after change to string basis.

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